TORSIONAL STIFFNESS OF ARBITRARILY SHAPED EMBEDDED FOUNDATIONS

By Shahid Ahmad¹ and George Gazetas,² Members, ASCE

ABSTRACT: A theoretical investigation of the static and dynamic torsional stiffness of arbitrarily shaped rigid foundations embedded in a homogeneous half-space, using a quadratic-element-based boundary element algorithm, is presented. Extensive parametric results are displayed in the form of dimensionless graphs for a variety of basement shapes and a wide range of embedment depths and types of contact between the foundation sidewalls and the surrounding soil. Based on these results, simple algebraic expressions are also developed, which can be readily used by engineers in practical applications. Discussion of the results provides insight into the nature of torsional response. A numerical example illustrates the practical applicability of the developed formulas, while a companion paper studies the torsional radiation damping of embedded foundations.

INTRODUCTION

Foundations are frequently designed to transmit static or dynamic torsional loads onto the supporting soil. Such loading arises, for instance, whenever asymmetric horizontal forces act on a superstructure, as may be the case during wind storms, earthquake shaking, operation of reciprocating engines, and horizontal movements of the antennas of radar towers. Torsional loading is also an important element in the design of critical power plants, so that they can withstand asymmetric tornado and impact forces. During earthquake shaking, even the vertically incident shear-wave components will indirectly generate torsion of the foundation of a nonsymmetric structure. Thus, predicting and limiting the resulting torsional rotation is a key geotechnical consideration, especially with a modern deformation-sensitive structure. Assessment of the static or dynamic response of a superstructure due to torsional loading involves estimation of torsional stiffness of its foundation system. Once the stiffness of the foundation-soil system is known, the response of the foundation or the structure supported on it can be easily evaluated.

A review of the literature reveals that a number of analytical, numerical, and simplified solutions for static and dynamic torsional stiffness of surface as well as fully embedded prismatic foundations have been reported [for a detailed review refer to Gazetas (1983)]. They include the work of Reissner and Sagoci (1944), Weissmann (1971), Kaldjian (1971), Novak and Sachs (1973), Veletsos and Nair (1974), Luco (1976), Kausel and Ushijima (1976), Day (1977), Dominguez (1978), Wong and Luco (1978), Constantinou and Gazetas (1984), and Veletsos and Dotson (1987). Recently, Wolf (1988) and Pais and Kausel (1988) have presented approximate formulas for static

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and dynamic stiffnesses of surface and fully embedded cylindrical and rectangular foundations by curve fitting their numerical results. On the other hand, Erden (1974), Stokoe and Richart (1974), and Novak (1985), among others, have presented the results of dynamic experimental investigations and have elucidated the importance of imperfect contact (due to sliding) between foundation sidewall and soil.

Most of the results on torsional response of embedded foundations published to date refer to cylindrical foundations, with the exceptions of Dominguez (1978), Dominguez and Abascal (1987), Pais and Kausel (1988), and Wolf (1988), who have studied fully embedded rectangular foundations with aspect ratios, \( L/B \) (i.e., length to width ratio), of 1 and 2. These solutions assume that the vertical sidewalls of the foundation and the surrounding soil are in perfect (welded) contact. In reality, however, net tensile stresses cannot be sustained at the soil-foundation interface, while shear tractions are limited by Coulomb’s friction law. Thus, as the aforementioned experiments have shown, separation and sliding are likely to occur, especially near the ground surface where the initial confining pressures are small.

In this paper, an accurate boundary element (BÉ) algorithm [see Ahmad et al. (1988) and Ahmad and Manolis (1987) for details] is used for a systematic parametric study of the torsional response of rigid foundations embedded in an elastic half-space. Results are presented for rectangular base shapes of aspect ratio, \( L/B \), up to 6, a wide range of embedment depths, with \( D/B \) [i.e., depth-to-width ratio; see Fig. 1(a)] up to 2, and the frequency range of usually greatest practical interests (normalized frequency, \( a_0 = 0.0 \) to 1.5). Results are also presented for circular, triangular, and T-shaped basements. Moreover, in every case, the height of perfect contact \( d \), between the vertical sidewalls and the surrounding soil is varied between the extremes of 0 and \( D \), the former representing foundations placed in an open trench, with no sidewall-backfill contact, and the latter corresponding to, the usually studied in dynamics but not necessarily more realistic, case of fully embed-

![Section](image)

**FIG. 1(a). Problem Geometry and Loading**

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ded foundations with perfect sidewall-backfill contact throughout the entire embedment depth. Since the torsional response of foundations is known to be independent of the Poisson’s ratio of the soil, a single value $v_s = 1/3$ is used in all analyses. Table 1 lists the cases studied in this parametric investigation.

The results are presented in the form of dimensionless graphs, and simple algebraic formulas are fitted to these numerical solutions. These formulas are also compared against available solutions from the aforementioned literature.

This paper, dealing with static and dynamic torsional stiffness, and the companion paper (Ahmad and Gazetas 1992) dealing with torsional radiation damping of embedded foundations, are a continuation of the work reported in Dobry and Gazetas (1986), Gazetas et al. (1985) Gazetas and Tassoulas (1987a, 1987b), Hatzikonstantinou et al. (1989), and Fotopoulos et al. (1989), dealing with the response of arbitrarily shaped surface and embedded foundations in vertical, horizontal, and rocking modes of vibration. However, this and the companion paper also differ from the listed works in two other aspects: (1) A more rigorous quadratic-element-based boundary-element formulation is now used instead of the simple constant-element-based BEM used in the earlier work; (2) for the dynamic torsional stiffness coefficient, in addition to the usual dimensionless graphs, frequency-dependent closed-form algebraic formulas are also developed for arbitrarily shaped embedded foundations [as was done earlier by Yeletsos

FIG. 1(b). Circular, Triangular, and Rectangular Basements with Circumscribed Rectangles
TABLE 1. List of Cases Studied Using Boundary Element Algorithm

<table>
<thead>
<tr>
<th>Basemat Shape (1)</th>
<th>Aspect ratio of circumscribed rectangle ( L/B ) (2)</th>
<th>Relative depth of embedment ( D/B ) (3)</th>
<th>Relative height of sidewall (over which sidewall is in full contact with soil) ( d/D ) (4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circular</td>
<td>1</td>
<td>0, 2/3, 4/3, 2</td>
<td>0, 1/2, 1</td>
</tr>
<tr>
<td>Triangular</td>
<td>1.15</td>
<td>0, 2</td>
<td>0, 1</td>
</tr>
<tr>
<td>T-shaped</td>
<td>1</td>
<td>0, 2</td>
<td>0, 1</td>
</tr>
<tr>
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<td>0, 1/8, 1/6, 1/5, 1/4, 1/3, 1/2, 2/3, 1, 4/3, 3/2, 2</td>
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</tr>
<tr>
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<td>0, 1/2, 1</td>
</tr>
<tr>
<td>Rectangular</td>
<td>3</td>
<td>0, 2/3, 4/3, 2</td>
<td>0, 1/2, 1</td>
</tr>
<tr>
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<td>0, 2/3, 4/3, 2</td>
<td>0, 1/2, 1</td>
</tr>
<tr>
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<td>5</td>
<td>0, 2</td>
<td>0, 1</td>
</tr>
<tr>
<td>Rectangular</td>
<td>6</td>
<td>0, 2/3, 4/3, 2</td>
<td>0, 1/2, 1</td>
</tr>
</tbody>
</table>

Note: For all cases, symmetric contact along perimeter and Poisson ratio of soil = 1/3 have been used.


PROBLEM STATEMENT

A massless rigid foundation, shown in Fig. 1(a), consists of a horizontally placed basemat of an arbitrary shape [hatched area in Fig. 1(a)] located at a depth \( D \) below the ground surface, and of vertical sidewalls having any degree of contact with the surrounding soil. A rectangle of dimensions \( 2L \) by \( 2B \) circumscribed to this arbitrary shape basemat is shown in Fig. 1(a), and Fig. 1(b) illustrates circumscribed rectangles for circular, triangular, and T-shaped basements. The foundation is subjected to steady-state vibration by a harmonic torsional moment about the z-axis, \( M_z = M_0 \exp(i\omega t) \), having amplitude \( M_0 \) and circular frequency \( \omega \). Static loading is treated as the special case of \( \omega = 0 \). The response of the foundation due to the applied torsional moment is a steady-state harmonic rotation, \( \theta = \theta_0 \exp(i\omega t) \), about the z-axis, having amplitude \( \theta_0 \). Due to the generated damping in the system, \( M_0 \) is generally out of phase with \( \theta \).

The soil is assumed to be a homogeneous elastic half-space—a choice stemming from the need to keep the number of independent problem parameters to a minimum, while trying to understand and quantify the role of basemat shape and of partial embedment. The soil is assumed to have no material damping; such damping can be readily incorporated by using the correspondence principle (Gazetas and Tassoulas 1987a, 1987b).

The foundation response is usually expressed through the dynamic moment-rotation ratio, known as foundation impedance. Using complex notation, the torsional impedance is expressed as follows:

\[
\kappa = \frac{M_z}{\theta_z} = \bar{K}_t + i\omega C_t \hfill \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)
\]
where $\bar{K}_r = \text{torsional dynamic spring constant (or stiffness)}$; $C_r = \text{torsional dashpot constant}$; and both are functions of circular frequency $\omega$.

The spring constant $\bar{K}_r$ reflects the contribution of stiffness and inertia of the soil, while the dashpot constant $C_r$ reflects the radiation damping due to waves spreading away from the foundation. It is convenient to express $\bar{K}_r$ (with overbar) as a product of the static stiffness $K_r$ (without overbar) times a dynamic spring coefficient, $k_r$ (a dimensionless parameter)

$$\bar{K}_r(a_o) = K_r \cdot k_r(a_o) \tag{2}$$

$\bar{K}_r$ and $k_r$ are functions of the dimensionless frequency $a_o = \omega B/V_s$, where $B = \text{the half-width of the foundation}$; and $V_s = \text{the shear wave velocity in the soil}$. This paper presents information related to $K_r$ and $k_r(a_o)$, while the companion paper (Ahmad and Gazetas 1992) studies $C_r(a_o)$.

**Boundary Element Analysis**

In recent years, the so-called boundary element method (BEM) has evolved as a very effective numerical technique for solving static and dynamic problems of three-dimensional foundations. Most of the existing BE-based work on foundation dynamics utilizes constant elements to model the geometry and represent the field variables (i.e., displacements and tractions are assumed constant over an element), whereas Kobayashi and Nishimura (1983), Ahmad (1986), Ahmad and Manolis (1987), and Ahmad and Banerjee (1988) have demonstrated the superiority of higher-order (quadratic or higher) boundary elements in achieving compatibility with the wavy nature of deformations associated with dynamic problems. However, to obtain results of acceptable accuracy with lower-order (linear or constant) elements, a fine mesh with a large number of elements may be necessary, especially for embedded foundations. Furthermore, a constant-element-based BE analysis cannot easily take into account the nonunique tractions at the corner and edges of the boundary (very important for embedded foundations); such effects are automatically handled in a formulation with quadratic boundary elements.

This study uses a frequency-domain direct boundary element formulation incorporating higher-order quadratic isoparametric elements, the fundamental solution for a homogeneous full space (dynamic Kelvin problem), and a self-adapting numerical integration scheme. The tractions at corners are computed by following Riccardella’s (1973) approach. This formulation has been used successfully by the first writer to study several soil dynamics problems (Ahmad 1986; Ahmad and Manolis 1987; Ahmad et al. 1988). Details on this formulation, approximations involved, discretization scheme, convergence, numerical integration scheme have been reported in the aforementioned publication by the first writer and colleagues.

A convergence study was conducted to obtain the optimum meshes for both surface and embedded foundations. For surface foundations with aspect ratio $L/B \leq 4$, it was found that free-surface discretization is not needed. Whereas, for surface foundations with aspect ratio $L/B > 4$, discretization of a small portion of the free surface in the neighborhood of the foundation, extending to $5–12B$ from the edge of the foundation, yields accurate results in the frequency range studied. For embedded foundations, the ground surface discretization needs to be extended further (12–20B) to account for waves emanating from the foundation sidewalls and impinging on the ground surface. Moreover, the foundation-soil contact area (at basement and side-
walls) and the adjacent ground surface (up to a distance 5B from foundation) need finer discretization (element length = 1/4–1/6 of shear wavelength); coarser elements with length of about one-half of the shear wavelength suffice for ground-surface areas beyond 5B.

**Static Stiffness of Surface Foundations**

For a surface foundation of arbitrary (but solid) shape, the torsional static stiffness \( K_{r,\text{surf}} \) is found to be a function of the polar moment of area of the basement \( I_b \), the shear modulus of soil \( G \), and a dimensionless parameter \( S_t \):

\[
K_{r,\text{surf}} = GI_b^{0.75}S_t \quad \text{(3a)}
\]

By trial and error, \( S_t \) is found to be uniquely related to two base-shape parameters: \( I_b/B^4 \) and \( A_r \), where for circular foundations \( A_r = 1.0 \), and for all other shapes \( A_r = 4B^2/I_b \).

\[
S_t = 3.48 \left( \frac{I_b}{B^4} \right)^{0.19} (A_r)^{0.41} \quad \text{(3b)}
\]

As depicted in Fig. 1, \( A_b \) = plan area of the actual foundation basement surface; \( 2L \) and \( 2B \) = length and width of the rectangle circumscribed to the basement \( (L > B) \); and \( I_b \) = the polar moment of contact area between the basement and the underlaying soil, about the vertical axis of rotation \( z \).

The dependence of \( S_t \) on \((I_b/B^4)\) and \(A_r\) is shown in Fig. 2, where the numerical data points are from the writers' boundary element analyses. Notice that the equivalent-circle approximation gives \( S_t = 3.8 \) for all shapes, thereby underpredicting appreciably the torsional stiffness of rectangular foundations with high aspect ratios. On the other hand, the foregoing simple algebraic expression (3) fits the numerical results for \( S_t \) very well.

**Effects of Embedment on Static Stiffness**

There are three possible effects of embedment on the torsional static stiffness of a rigid foundation. First, in practice, a foundation mat placed

![FIG. 2. Torsional Static Stiffness Factor, \( S_t \)](image-url)
at a depth $D$ below the ground surface transmits the load to a soil that is deeper and, therefore, different (usually stiffer) than the soil affected by a similar surface foundation. Thus, other things being equal, a greater depth $D$ implies, in general, different (usually greater) stiffness. This important effect is not addressed further herein. However, when applying the proposed formulas to a practical situation, the engineer must select a representative value of soil modulus for the particular depth of embedment.

The other two effects that modify the stiffness of embedded foundations are addressed in this paper and are referred to as the trench effect and the sidewall-contact effect, using the terminology of Gazetas and Tassoulas (1987a, 1987b).

With reference to Fig. 3, the trench effect stems from the fact that, even in a perfectly homogeneous half-space, the torsional rotation of a foundation placed at the bottom of an open trench is smaller than that of the same foundation on the ground surface. To understand why, visualize a horizontal plane surface passing through the base. In case of the surface foundation, this plane deforms free of any external stress, while for the embedded foundation, normal and shear stresses from the overlying soil seem to restrict its movement (see Fig. 3), thereby increasing the foundation stiffness from $K_{t,surf}$ to $K_{t,tren} = \Gamma_{t,tren} K_{t,surf}$. In addition to the numerical data of the

![Diagram](image)

**FIG. 3.** Schematic Illustration of Effects of Embedment on Torsional Static Stiffness: (a) Surface Foundation; (b) Foundation Placed in Open Trench; and (c) Embedded Foundation
present investigation, Erden’s (1974) experimental work also supports this outcome.

The trench factor $\Gamma_{t,tren}$, as shown in Fig. 4, is found by trial and error to be uniquely related to the parameters $(D/B)$ and $(I_b/B^4)$ for all foundation shapes:

$$\Gamma_{t,tren} = \frac{K_{t,tren}}{K_{t,surf}} = 1 + 0.5 \left(\frac{D}{B}\right)^{0.1} \left(\frac{I_b}{B^4}\right)^{-0.13} \hfill (4)$$

The sidewall-contact effect is more obvious arising from the fact that part of the applied torsional moment is transmitted into the ground through shear and normal tractions acting on the vertical sidewalls that are in contact with the surrounding soil. These additional transmission paths lead to a

![Graph showing Trench Effect on Torsional Static Stiffness](image1)

**FIG. 4. Trench Effect on Torsional Static Stiffness**

![Graph showing Sidewall Contact Effect on Torsional Static Stiffness](image2)

**FIG. 5. Sidewall Contact Effect on Torsional Static Stiffness**
further increase in the static stiffness of an embedded foundation compared
to that in an open trench, by a factor $\Gamma_{t,\text{wall}}$.

Careful examination of the numerical data points shown in Fig. 5 leads to
the following conclusions:

1. Sidewall-contact factor $\Gamma_{t,\text{wall}}$ may reach high values, depending on the
degree of sidewall-soil contact and foundation geometry.
2. For a given basemat, $\Gamma_{t,\text{wall}}$ depends on $d$, $D$, $B$, and $L$ in a complex
manner. By trial and error, an excellent correlation is unveiled between $\Gamma_{t,\text{wall}}$
and the dimensionless parameters $L/B$, $d/D$, $d/B$, and $I_w/I_b$. Indeed, the scatter
in Fig. 5 of the numerical (BE) data points around the curve

$$
\Gamma_{t,\text{wall}} = \frac{K_{r,\text{emb}}}{K_{r,\text{tren}}} = 1 + 0.38 \left( \frac{I_w}{I_b} \right) \left( \frac{d}{D} \right)^{-0.51} \left( \frac{D}{B} \right)^{-0.6} \left( \frac{L}{B} \right)^{0.06} \quad \quad \quad (5)
$$

is negligibly small.

In (5), $I_w$ is the sum of the second moment of areas of vertical sidewalls
about the axis of rotation, $z$. For circular and rectangular basemats, the
dimensionless parameter $I_w/I_b$ is

$$
\frac{I_w}{I_b} = \frac{d(B^3 + L^2 + 3BL^2 + 3B^2L)}{BL(B^2 + L^2)} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
### TABLE 2. Torsional Static Stiffness of Fully Embedded Foundations

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<td>—</td>
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<td>—</td>
<td>—</td>
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<td>29.7</td>
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<td>—</td>
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<td>69.70</td>
<td>64.41</td>
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<td>—</td>
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<tr>
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<td>126.00</td>
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<td>136.20</td>
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<td>Rectangular, $L/B = 4$</td>
<td>2/3</td>
<td>263.53</td>
<td>256.20</td>
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<td>348.46</td>
<td>343.50</td>
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<tr>
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<td>2</td>
<td>429.60</td>
<td>422.80</td>
<td>—</td>
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<tr>
<td>Rectangular, $L/B = 6$</td>
<td>0</td>
<td>351.20</td>
<td>354.80</td>
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<td>626.80</td>
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basemat \( I_i/B^4 \). By trial and error, a good correlation is found between \( k_i \) and the aforementioned parameters, expressed algebraically in the form

\[
(k_i)_{\text{surf}} = 1 - \frac{A_1(a_o)^{1.8}}{1 + A_2(a_o)^2} \tag{8a}
\]

where

\[
A_1 = 0.08 \left( 2 + \frac{I_b}{B^4} \right)^{0.64} \tag{8b}
\]

\[
A_2 = 0.3 + 0.75 \left( \frac{L}{B} - 1 \right)^2 \tag{8c}
\]

Fig. 6 displays, for various basemat shapes, both the numerical results by the BE analysis and the corresponding curves from (8). The following trends deserve a note:

1. \( k_i \) decreases with increasing values of \( a_o \) for basemat shapes having \( L/B \leq 2 \), in the frequency range studied.

2. The results for the T-shaped and circular foundations, both having the circumscribed rectangle of aspect ratio \( L/B = 1 \), practically coincide with those for the square foundation. This shows that, for the various basemat shapes studied, the exact shape of foundation mats that have the same \( L/B \) does not matter in practical terms.

**Foundation Placed in Open Trench \((d = 0)\)**

The change that occurs in the dynamic spring coefficients when a foundation is placed at the bottom of a trench without sidewalls is a function of two dimensionless parameters, the dimensionless frequency \( a_o \), and the relative depth of embedment \( D/B \). Fig. 7 portrays the effect of \( D/B \) for a rectangular foundation mat \((L/B = 2)\). The values of \( k_i \) for foundations in a trench seem to exceed the corresponding values for the square foundation, with the difference increasing with \( a_o \). A modification factor \( f_{\text{trench}} \) can be incorporated into (8), to account for the trench effect in the dynamic spring coefficient, as follows:

\[
(k_i)_{\text{trench}} = 1 - \frac{A_1(a_o)^{1.8}}{1 + A_2(a_o)^2} f_{\text{trench}} \tag{9a}
\]

where \( A_1 \) and \( A_2 \) remain the same as in (8), and

\[
f_{\text{trench}} = 1 - 0.25 \left( \frac{D}{B} \right)^{0.12} (a_o)^{1.2} \tag{9b}
\]

Fig. 8 displays the numerical BE data points for foundations of various basemat shapes with \( D/B = 2 \), and the corresponding curves obtained by (9).

**Partially and Fully Embedded Foundations**

The effect of partial embedment, expressed through \( d/D \), on the dynamic spring coefficients of a rectangular foundation \((L/B = 2, D/B = 2/3)\) is shown in Fig. 9. Results are presented for three values of \( d/D \) (ratio between
the height of sidewall in contact with the surrounding soil and the depth of the trench). Note that, $d/D = 0$ represents the case of a foundation with no sidewall-soil contact, $d/D = 1$ represents the fully embedded case, while $d/D = 0.5$ is an intermediate case. It can be seen that an increase in the sidewall-soil contact of a foundation results in a decrease in the values of dynamic spring coefficients. Moreover, it should be noted that the rate of decrease for $k$, declines significantly with increasing values of $d/D$. However, these arguments should not be extended to dynamic stiffness, $K_d$. As discussed earlier, $K_d = K_s \cdot k_s$, where $K_s$ is the static stiffness that increases with the height of sidewall contact $d$; hence, a fully embedded foundation would generally have higher stiffness. The sidewall-contact effect on the dynamic spring coefficients is found to depend on the dimensionless frequency $a_o$ and a dimensionless sidewall-contact parameter $d/L$. A modification factor
$f_{\text{wall}}$ can be incorporated in (9) to account for the effect of sidewall contact as follows:

$$k_i = 1 - \frac{A_1(a_o)^{1.8}}{1 + A_2(a_o)^2} f_{\text{tren}} f_{\text{wall}} \quad \cdots \quad (10a)$$

where

$$f_{\text{wall}} = 1 + \left( \frac{d}{L} \right)^{0.32} (a_o)^{-0.1} \quad \cdots \quad (10b)$$

A comparison of the foregoing formula [(10)] and the numerical (BEM) results for fully embedded foundations is displayed in Fig. 10. Notice that a reasonable agreement exists between the two sets of results.

Finally, dynamic spring coefficients of circular, square, and a few rectangular surface foundations obtained using the proposed formula [(10)] are presented in Table 3, with the BE results and the published solutions of several researchers. The proposed formula seems to agree reasonably well with rigorous solutions.

**ILLUSTRATIVE EXAMPLE**

The use of the simple formulas presented is demonstrated herein in estimating the dynamic stiffness $K$, of an embedded foundation, shown in Fig. 11. The basemat is a truncated rectangle, placed 8 m below the ground surface. The circumscribed rectangle has the dimensions, 45 m $\times$ 15 m. A 6 m-high sidewall, in perfect contact with the surrounding soil, is built around the periphery. The excitation frequency, $o$, is equal to 10 rad/s.

Using the geometric parameters listed in Fig. 11, the static stiffness $K_i$ is computed first

$$K_{i,\text{surf}} = 3.48 \ G I_b^{0.75} \left( \frac{I_b}{B^4} \right)^{0.19} \left( \frac{4B^2}{A_b} \right)^{0.41} \quad \cdots \quad (11a)$$
FIG. 9. Embedded Foundation: Effect of Partial Embedment on Dynamic Spring Coefficient, $k_t$

FIG. 10. Dynamic Spring Coefficient, $k_t$, of Embedded Foundations with Full Sidewall-Soil Contact ($d/D = 1$)

$$K_{t, \text{surf}} = (3.48)(68)(14308)^{0.75}(32.97)^{0.19}(0.3644)^{0.41}$$  
$$K_{t, \text{surf}} = 18 \times 10^5 \text{ MN.m}$$

$$\Gamma_{t, \text{tren}} = 1.0 + 0.5 \left( \frac{D}{B} \right)^{0.1} \left( \frac{L_b}{B^4} \right)^{-0.13}$$

$$\Gamma_{t, \text{tren}} = 1.0 + (0.5)(1.07)^{0.1}(32.97)^{-0.13} = 1.32$$

$$\Gamma_{r, \text{wall}} = 1.0 + 0.38 \left( \frac{l_w}{l_b} \right) \left( \frac{d}{D} \right)^{-0.51} \left( \frac{D}{B} \right)^{-0.6} \left( \frac{L}{B} \right)^{0.06}$$

$$\Gamma_{r, \text{wall}} = 1.0 + (0.38)(1.236)(0.75)^{-0.51}(1.07)^{-0.6}(3)^{0.06}$$

$$\Gamma_{r, \text{wall}} = 1.56$$
<table>
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<tr>
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<td>Circular</td>
<td>0.3</td>
<td>0.980</td>
<td>0.983</td>
<td>0.983</td>
<td>0.966</td>
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<tr>
<td>Circular</td>
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<td>0.935</td>
<td>0.938</td>
<td>0.938</td>
<td>0.907</td>
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<td>0.883</td>
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<td>0.825</td>
<td>0.828</td>
<td>0.829</td>
<td>0.793</td>
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<td>0.820</td>
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<tr>
<td>Circular</td>
<td>1.5</td>
<td>0.776</td>
<td>0.780</td>
<td>0.783</td>
<td>0.758</td>
<td>-</td>
<td>0.761</td>
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<tr>
<td>Square</td>
<td>0.3</td>
<td>0.976</td>
<td>0.984</td>
<td>0.976</td>
<td>0.967</td>
<td>0.964</td>
<td>-</td>
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<tr>
<td>Square</td>
<td>0.6</td>
<td>0.923</td>
<td>0.929</td>
<td>0.918</td>
<td>0.897</td>
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<tr>
<td>Square</td>
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<td>0.857</td>
<td>0.868</td>
<td>0.851</td>
<td>0.834</td>
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<tr>
<td>Square</td>
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<td>0.792</td>
<td>0.813</td>
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<td>0.787</td>
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<tr>
<td>Square</td>
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<td>0.734</td>
<td>0.749</td>
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<td>0.3</td>
<td>0.952</td>
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<td>0.961</td>
<td>0.947</td>
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<td>0.867</td>
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<td>Rectangular, $L/B = 2$</td>
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<td>0.795</td>
<td>0.790</td>
<td>0.817</td>
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<td>0.788</td>
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<td>0.745</td>
<td>0.795</td>
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<td>Rectangular, $L/B = 2$</td>
<td>1.5</td>
<td>0.717</td>
<td>0.731</td>
<td>0.783</td>
<td>0.763</td>
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<td>0.3</td>
<td>0.898</td>
<td>0.885</td>
<td>0.894</td>
<td>0.949</td>
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<td>Rectangular, $L/B = 4$</td>
<td>0.6</td>
<td>0.836</td>
<td>0.826</td>
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<td>0.869</td>
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<tr>
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<td>0.9</td>
<td>0.821</td>
<td>0.825</td>
<td>0.795</td>
<td>0.814</td>
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<tr>
<td>Rectangular, $L/B = 4$</td>
<td>1.2</td>
<td>0.819</td>
<td>0.826</td>
<td>0.784</td>
<td>0.783</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Rectangular, $L/B = 4$</td>
<td>1.5</td>
<td>0.821</td>
<td>0.835</td>
<td>0.778</td>
<td>0.764</td>
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Thus, the torsional static stiffness of the given embedded foundation is

\[ K_{r,\text{emb}} = K_{r,\text{surf}} \Gamma_{r,\text{trench}} \Gamma_{r,\text{wall}} \] ................................. (14a)

\[ K_{r,\text{emb}} = (18 \times 10^5)(1.32)(1.56) = 37 \times 10^5 \text{ MN.m} \] ........................ (14b)

The next step involves the calculation of dynamic spring coefficient, \( k_\omega \), at

\[ a_\omega = \omega B/V_0 = 10 \times 7.5/194 = 0.39 \]. To this end, the variables appearing in (10), are [from (8) and (9)]

\[ A_1 = 0.08 \left( 2 + \frac{I_b}{B^4} \right)^{0.64} = 0.08(2 + 32.97)^{0.64} = 0.78 \] ........... (15)

\[ A_2 = 0.3 + 0.75 \left( \frac{L}{B} - 1 \right)^2 = 0.3 + (0.75)(3 - 1)^2 = 3.3 \] ........ (16)
\[ f_{\text{tren}} = 1 - 0.25 \left( \frac{D}{B} \right)^{0.32} (a_o)^{1.2} = 1 - (0.25)(1.07)^{0.12}(0.39)^{1.2} = 0.92 \]  

\[ f_{\text{wall}} = 1 + \left( \frac{d}{L} \right)^{0.32} (a_o)^{-0.1} = 1 + (0.133)^{0.32}(0.39)^{-0.1} = 1.58 \ldots \]  

Therefore

\[ k_t = 1 - \frac{A_1(a_o)^{1.8}}{1 + A_2(a_o)^{1.2}} f_{\text{tren}} f_{\text{wall}} \]  

\[ k_t = 1 - \frac{0.78(1.35)^{1.8}}{1 + 3.3(1.35)^2} (0.92)(1.58) = 1.2 \]  

Hence, the torsional dynamic stiffness \( K_t \) is obtained by multiplying \( K_t \) by \( k_t \)

\[ K_t = K_t k_t = (37 \times 10^5)(1.2) = 45 \times 10^5 \text{ MN.m} \]

**CONCLUSION**

This paper presents a comprehensive set of parametric numerical results obtained with a rigorous boundary element formulation, on the basis of which it develops closed-form simple algebraic formulas for estimating inexpensively and reliably the static and dynamic torsional stiffness of rigid foundations embedded in relatively deep homogeneous soil deposits. These formulas are applicable to foundations having any arbitrary (but solid) base-mat shape and various degrees of contact between the vertical sidewalls and the surrounding soil: complete-and-perfect contact, partial symmetric and nonsymmetric contact, and no contact at all. The practical applicability of the developed formulas is illustrated with a numerical example, while a companion paper (Ahmad and Gazetas, 1992) extends this study to radiation damping.

**APPENDIX I. REFERENCES**


